

# INTERIOR AND EXTERIOR DIFFERENTIAL SYSTEMS FOR LIE ALGEBROIDS

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## Abstract

A theorem of Maurer-Cartan type and two theorems of Cartan type are presented.

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## 1 Introduction

The motivation for our researches was to extend the classical notion of exterior differential systems (see: [2, 4, 5, 6]) for Lie algebroids.

For the first time, we introduced the notion of *interior differential system* (IDS) of a Lie algebroid, and, using the exterior differential calculus for Lie algebroids, (see: [3, 7]) we establish the structure equations of Maurer-Cartan type and we characterize the involutivity of an IDS in a theorem of Cartan type.[1] Finally, using the notion of *exterior differential system* (EDS) of a Lie algebroid, we characterize the involutivity of an IDS in a theorem of Cartan type.[1] In particular, we obtain similar results with classical results.

## 2 Preliminaries

In general, if  $\mathcal{C}$  is a category, then we denoted by  $|\mathcal{C}|$  the class of objects and for any  $A, B \in |\mathcal{C}|$ , we denote by  $\mathcal{C}(A, B)$  the set of morphisms of  $A$  source and  $B$  target.

Let **Liealg**, **Mod**, and  $\mathbf{B}^v$  be the category of Lie algebras, modules and vector bundles respectively.

We know that if  $(E, \pi, M) \in |\mathbf{B}^v|$ , then  $(\Gamma(E, \pi, M), +, \cdot)$  is a  $\mathcal{F}(M)$ -module.

In addition, if  $(E, \pi, M) \in |\mathbf{B}^v|$  such that  $M$  is paracompact and if  $A \subseteq M$  is closed, then for any section  $u$  over  $A$  it exists  $\tilde{u} \in \Gamma(E, \pi, M)$  such that  $\tilde{u}|_A = u$ .

**Note:** In the following, we consider only vector bundles with paracompact base.

We know that a Lie algebroid is a vector bundle  $(F, \nu, N) \in |\mathbf{B}^v|$  such that there exists

$$(\rho, Id_N) \in \mathbf{B}^v((F, \nu, N), (TN, \tau_N, N))$$

and an operation

$$\begin{array}{ccc} \Gamma(F, \nu, N) \times \Gamma(F, \nu, N) & \xrightarrow{[\cdot]_F} & \Gamma(F, \nu, N) \\ (u, v) & \longmapsto & [u, v]_F \end{array}$$

with the following properties:

$LA_1$ . the equality holds good

$$[u, f \cdot v]_F = f [u, v]_F + \Gamma(\rho, Id_N)(u) f \cdot v,$$

for all  $u, v \in \Gamma(F, \nu, N)$  and  $f \in \mathcal{F}(N)$ ,

$LA_2$ . the 4-tuple  $(\Gamma(F, \nu, N), +, \cdot, [\cdot]_F)$  is a Lie  $\mathcal{F}(N)$ -algebra,

$LA_3$ . the **Mod**-morphism  $\Gamma(\rho, Id_N)$  is a **LieAlg**-morphism of  $(\Gamma(F, \nu, N), +, \cdot, [\cdot]_F)$  source and  $(\Gamma(TN, \tau_N, N), +, \cdot, [\cdot]_{TN})$  target.

Let  $\left((F, \nu, N), [\cdot]_F, (\rho, Id_N)\right)$  be a Lie algebroid.

- Locally, for any  $\alpha, \beta \in \overline{1, p}$ , we set  $[t_\alpha, t_\beta]_F \stackrel{put}{=} L_{\alpha\beta}^\gamma t_\gamma$ . We easily obtain that  $L_{\alpha\beta}^\gamma = -L_{\beta\alpha}^\gamma$ , for any  $\alpha, \beta, \gamma \in \overline{1, p}$ .

The real local functions  $\left\{L_{\alpha\beta}^\gamma, \alpha, \beta, \gamma \in \overline{1, p}\right\}$  will be called the *structure functions*.

- We assume that  $(F, \nu, N)$  is a vector bundle with type fibre the real vector space  $(\mathbb{R}^p, +, \cdot)$  and structure group a Lie subgroup of  $(\mathbf{GL}(p, \mathbb{R}), \cdot)$ . We take  $(x^i, z^\alpha)$  as canonical local coordinates on  $(F, \nu, N)$ , where  $i \in \overline{1, n}$ ,  $\alpha \in \overline{1, p}$ .

Consider

$$(x^i, z^\alpha) \longrightarrow (x^{i'}, z^{\alpha'})$$

a change of coordinates on  $(F, \nu, N)$ . Then the coordinates  $z^\alpha$  change to  $z^{\alpha'}$  by the rule:

$$(2.1) \quad z^{\alpha'} = \Lambda_\alpha^{\alpha'} z^\alpha.$$

- If  $z^\alpha t_\alpha \in \Gamma(F, \nu, N)$  is arbitrary, then

$$(2.2) \quad [\Gamma(\rho, Id_N)(z^\alpha t_\alpha)f](x) = \left(\rho_\alpha^i z^\alpha \frac{\partial f}{\partial x^i}\right)(x)$$

for any  $f \in \mathcal{F}(N)$  and  $x \in N$ .

The coefficients  $\rho_\alpha^i$  change to  $\rho_{\check{\alpha}}^{\check{i}}$  by the rule:

$$(2.3) \quad \rho_{\check{\alpha}}^{\check{i}} = \Lambda_\alpha^\alpha \rho_\alpha^i \frac{\partial x^{\check{i}}}{\partial x^i},$$

where

$$\|\Lambda_\alpha^\alpha\| = \left\| \Lambda_\alpha^\alpha \right\|^{-1}.$$

*Remark 2.1* The following equalities hold good:

$$(2.4) \quad \left(\rho_\alpha^i \frac{\partial}{\partial x^i}\right)(f) = \rho_\alpha^i \frac{\partial f}{\partial x^i}, \forall f \in \mathcal{F}(N).$$

and

$$(2.5) \quad L_{\alpha\beta}^\gamma \cdot \rho_\gamma^k = \rho_\alpha^i \cdot \frac{\partial \rho_\beta^k}{\partial x^i} - \rho_\beta^j \cdot \frac{\partial \rho_\alpha^k}{\partial x^j}.$$

### 3 Interior Differential Systems

Let  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$  be a Lie algebroid.

**Definition 3.1** Any vector subbundle  $(E, \pi, N)$  of the vector bundle  $(F, \nu, N)$  will be called *interior differential system (IDS) of the Lie algebroid*

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N)).$$

*Remark 3.1* If  $(E, \pi, N)$  is an IDS of the Lie algebroid

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N)),$$

then we obtain a vector subbundle  $(E^0, \pi^0, N)$  of the dual vector bundle  $\left(\overset{*}{F}, \overset{*}{\nu}, N\right)$  such that

$$\Gamma(E^0, \pi^0, N) \stackrel{put}{=} \left\{ \Omega \in \Gamma\left(\overset{*}{F}, \overset{*}{\nu}, N\right) : \Omega(S) = 0, \forall S \in \Gamma(E, \pi, N) \right\}.$$

The vector subbundle  $(E^0, \pi^0, N)$  will be called *the annihilator vector subbundle of the IDS  $(E, \pi, N)$* .

**Proposition 3.1** If  $(E, \pi, N)$  is an IDS of the Lie algebroid

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$$

such that  $\Gamma(E, \pi, N) = \langle S_1, \dots, S_r \rangle$ , then it exists  $\Theta^{r+1}, \dots, \Theta^p \in \Gamma\left(\overset{*}{F}, \overset{*}{\nu}, N\right)$  linearly independent such that  $\Gamma(E^0, \pi^0, N) = \langle \Theta^{r+1}, \dots, \Theta^p \rangle$ .

**Definition 3.2** The IDS  $(E, \pi, N)$  of the Lie algebroid

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$$

will be called *involutive* if  $[S, T]_F \in \Gamma(E, \pi, N)$ , for any  $S, T \in \Gamma(E, \pi, N)$ .

**Proposition 3.2** If  $(E, \pi, N)$  is an IDS of the Lie algebroid

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, \eta))$$

and  $\{S_1, \dots, S_r\}$  is a base for the  $\mathcal{F}(M)$ -submodule  $(\Gamma(E, \pi, N), +, \cdot)$  then  $(E, \pi, N)$  is involutive if and only if  $[S_a, S_b]_F \in \Gamma(E, \pi, N)$ , for any  $a, b \in \overline{1, r}$ .

## 4 Exterior differential calculus

Let  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$  be a Lie algebroid.

We denoted by  $\Lambda^q(F, \nu, N)$  the set of *differential forms of degree  $q$* . If

$$\Lambda(F, \nu, N) = \bigoplus_{q \geq 0} \Lambda^q(F, \nu, N),$$

then we obtain the *exterior differential algebra*  $(\Lambda(F, \nu, N), +, \cdot, \wedge)$ .

**Definition 4.1** For any  $z \in \Gamma(F, \nu, N)$ , the application

$$\Lambda(F, \nu, N) \xrightarrow{L_z} \Lambda(F, \nu, N),$$

defined by

$$L_z(f) = [\Gamma(\rho, Id_N) z](f),$$

for any  $f \in \mathcal{F}(N)$  and

$$\begin{aligned} L_z \omega(z_1, \dots, z_q) &= [\Gamma(\rho, Id_N) z](\omega((z_1, \dots, z_q))) \\ &\quad - \sum_{i=1}^q \omega((z_1, \dots, [z, z_i]_F, \dots, z_q)), \end{aligned}$$

for any  $\omega \in \Lambda^q(F, \nu, N)$  and  $z_1, \dots, z_q \in \Gamma(F, \nu, N)$ , is called the *covariant Lie derivative with respect to the section  $z$* .

**Theorem 4.1** If  $z \in \Gamma(F, \nu, N)$ ,  $\omega \in \Lambda^q(F, \nu, N)$  and  $\theta \in \Lambda^r(F, \nu, N)$ , then

$$(4.1) \quad L_z(\omega \wedge \theta) = L_z \omega \wedge \theta + \omega \wedge L_z \theta.$$

**Definition 4.2** For any  $z \in \Gamma(F, \nu, N)$ , the application

$$\begin{aligned} \Lambda(F, \nu, N) &\xrightarrow{i_z} \Lambda(F, \nu, N) \\ \Lambda^q(F, \nu, N) \ni \omega &\longmapsto i_z \omega \in \Lambda^{q-1}(F, \nu, N), \end{aligned}$$

defined by  $i_z f = 0$ , for any  $f \in \mathcal{F}(N)$  and

$$i_z \omega(z_2, \dots, z_q) = \omega(z, z_2, \dots, z_q),$$

for any  $z_2, \dots, z_q \in \Gamma(F, \nu, N)$ , is called the *interior product associated to the section  $z$* .

**Theorem 4.2** If  $z \in \Gamma(F, \nu, N)$ , then for any  $\omega \in \Lambda^q(F, \nu, N)$  and  $\theta \in \Lambda^r(F, \nu, N)$  we obtain

$$(4.2) \quad i_z(\omega \wedge \theta) = i_z \omega \wedge \theta + (-1)^q \omega \wedge i_z \theta.$$

**Theorem 4.3** For any  $z, v \in \Gamma(F, \nu, N)$  we obtain

$$(4.3) \quad L_v \circ i_z - i_z \circ L_v = i_{[z,v]_F}.$$

**Theorem 4.4** The application

$$\begin{array}{ccc} \Lambda^q(F, \nu, N) & \xrightarrow{d^F} & \Lambda^{q+1}(F, \nu, N) \\ \omega & \longmapsto & d\omega \end{array}$$

defined by

$$d^F f(z) = \Gamma(\rho, Id_N)(z) f,$$

for any  $z \in \Gamma(F, \nu, N)$ , and

$$\begin{aligned} d^F \omega(z_0, z_1, \dots, z_q) &= \sum_{i=0}^q (-1)^i \Gamma(\rho, Id_N) z_i (\omega((z_0, z_1, \dots, \hat{z}_i, \dots, z_q))) \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([z_i, z_j]_F, z_0, z_1, \dots, \hat{z}_i, \dots, \hat{z}_j, \dots, z_q), \end{aligned}$$

for any  $z_0, z_1, \dots, z_q \in \Gamma(F, \nu, N)$ , is unique with the following property:

$$(4.4) \quad L_z = d^F \circ i_z + i_z \circ d^F, \quad \forall z \in \Gamma(F, \nu, N).$$

This application is called *the exterior differentiation operator for the exterior differential algebra of the Lie algebroid  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$* .

**Theorem 4.5** The exterior differentiation operator  $d^F$  given by the previous theorem has the following properties:

1. For any  $\omega \in \Lambda^q(F, \nu, N)$  and  $\theta \in \Lambda^r(F, \nu, N)$  we obtain

$$(4.5) \quad d^F(\omega \wedge \theta) = d^F \omega \wedge \theta + (-1)^q \omega \wedge d^F \theta.$$

2. For any  $z \in \Gamma(F, \nu, N)$  we obtain

$$(4.6) \quad L_z \circ d^F = d^F \circ L_z.$$

3.  $d^F \circ d^F = 0$ .

**Theorem 4.6** (of Maurer-Cartan type)

If  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$  is a Lie algebroid and  $d^F$  is the exterior differentiation operator for the exterior differential  $\mathcal{F}(N)$ -algebra  $(\Lambda(F, \nu, N), +, \cdot, \wedge)$ , then we obtain the structure equations of Maurer-Cartan type

$$(C_1) \quad d^F t^\alpha = -\frac{1}{2} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma, \quad \alpha \in \overline{1, p}$$

and

$$(C_2) \quad d^F x^i = \rho_\alpha^i t^\alpha, \quad i \in \overline{1, n},$$

where  $\{t^\alpha, \alpha \in \overline{1, p}\}$  is the coframe of the vector bundle  $(F, \nu, N)$ .

This equations will be called *the structure equations of Maurer-Cartan type associated to the Lie algebroid  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$* .

*Proof.* Let  $\alpha \in \overline{1, p}$  be arbitrary. Since

$$d^F t^\alpha (t_\beta, t_\gamma) = -L_{\beta\gamma}^\alpha, \quad \forall \beta, \gamma \in \overline{1, p}$$

it results that

$$(1) \quad d^F t^\alpha = - \sum_{\beta < \gamma} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma.$$

Since  $L_{\beta\gamma}^\alpha = -L_{\gamma\beta}^\alpha$  and  $t^\beta \wedge t^\gamma = -t^\gamma \wedge t^\beta$ , for nay  $\beta, \gamma \in \overline{1, p}$ , it results that

$$(2) \quad \sum_{\beta < \gamma} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma = \frac{1}{2} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma$$

Using the equalities (1) and (2) it results the structure equation  $(\mathcal{C}_1)$ .

Let  $i \in \overline{1, n}$  be arbitrary. Since

$$d^F x^i (t_\alpha) = \rho_\alpha^i, \quad \forall \alpha \in \overline{1, p}$$

it results the structure equation  $(\mathcal{C}_2)$ .

*q.e.d.*

**Theorem 4.7** (of Cartan type) *Let  $(E, \pi, N)$  be an IDS of the Lie algebroid*

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N)).$$

*If  $\{\Theta^{r+1}, \dots, \Theta^p\}$  is a base for the  $\mathcal{F}(N)$ -submodule  $(\Gamma(E^0, \pi^0, N), +, \cdot)$ , then the IDS  $(E, \pi, N)$  is involutive if and only if it exists*

$$\Omega_\beta^\alpha \in \Lambda^1(F, \nu, N), \quad \alpha, \beta \in \overline{r+1, p}$$

*such that*

$$d^{h^*F} \Theta^\alpha = \sum_{\beta \in \overline{r+1, p}} \Omega_\beta^\alpha \wedge \Theta^\beta \in \mathcal{I}(\Gamma(E^0, \pi^0, N)).$$

*Proof.* Let  $\{S_1, \dots, S_r\}$  be a base for the  $\mathcal{F}(N)$ -submodule  $(\Gamma(E, \pi, N), +, \cdot)$

Let  $\{S_{r+1}, \dots, S_p\} \in \Gamma(F, \nu, N)$  such that  $\{S_1, \dots, S_r, S_{r+1}, \dots, S_p\}$  is a base for the  $\mathcal{F}(N)$ -module

$$(\Gamma(F, \nu, N), +, \cdot).$$

Let  $\Theta^1, \dots, \Theta^r \in \Gamma\left(\overset{*}{F}, \overset{*}{\nu}, N\right)$  such that  $\{\Theta^1, \dots, \Theta^r, \Theta^{r+1}, \dots, \Theta^p\}$  is a base for the  $\mathcal{F}(N)$ -module

$$\left(\Gamma\left(\overset{*}{F}, \overset{*}{\nu}, N\right), +, \cdot\right).$$

For any  $a, b \in \overline{1, r}$  and  $\alpha, \beta \in \overline{r+1, p}$ , we have the equalities:

$$\begin{aligned} \Theta^a(S_b) &= \delta_b^a \\ \Theta^a(S_\beta) &= 0 \\ \Theta^\alpha(S_b) &= 0 \\ \Theta^\alpha(S_\beta) &= \delta_\beta^\alpha \end{aligned}$$

We remark that the set of the 2-forms

$$\left\{ \Theta^a \wedge \Theta^b, \Theta^a \wedge \Theta^\beta, \Theta^\alpha \wedge \Theta^\beta, \quad a, b \in \overline{1, r} \wedge \alpha, \beta \in \overline{r+1, p} \right\}$$

is a base for the  $\mathcal{F}(M)$ -module  $(\Lambda^2(F, \nu, N), +, \cdot)$ .

Therefore, we have

$$(1) \quad d^F \Theta^\alpha = \sum_{b < c} A_{bc}^\alpha \Theta^b \wedge \Theta^c + \sum_{b, \gamma} B_{b\gamma}^\alpha \Theta^b \wedge \Theta^\gamma + \sum_{\beta < \gamma} C_{\beta\gamma}^\alpha \Theta^\beta \wedge \Theta^\gamma,$$

where,  $A_{bc}^\alpha, B_{b\gamma}^\alpha$  and  $C_{\beta\gamma}^\alpha$ ,  $a, b, c \in \overline{1, r}$ ,  $\alpha, \beta, \gamma \in \overline{r+1, p}$  are real local functions such that  $A_{bc}^\alpha = -A_{cb}^\alpha$  and  $C_{\beta\gamma}^\alpha = -C_{\gamma\beta}^\alpha$ .

Using the formula

$$(2) \quad \begin{aligned} d^F \Theta^\alpha(S_b, S_c) &= \Gamma(\rho, Id_N) S_b(\Theta^\alpha(S_c)) - \Gamma(\rho, Id_N) S_c(\Theta^\alpha(S_b)) \\ &\quad - \Theta^\alpha([S_b, S_c]_F), \end{aligned}$$

we obtain that

$$(3) \quad A_{bc}^\alpha = -\Theta^\alpha([S_b, S_c]_F),$$

for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ .

We admit that  $(E, \pi, N)$  is an involutive IDS of the Lie algebroid  $((F, \nu, N), [, ]_F, (\rho, Id_N))$ .

As  $[S_b, S_c]_F \in \Gamma(E, \pi, N)$ , for any  $b, c \in \overline{1, r}$ , it results that  $\Theta^\alpha([S_b, S_c]_F) = 0$ , for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ . Therefore, for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ , we obtain  $A_{bc}^\alpha = 0$  and

$$\begin{aligned} d^F \Theta^\alpha &= \sum_{b, \gamma} B_{b\gamma}^\alpha \Theta^b \wedge \Theta^\gamma + \frac{1}{2} C_{\beta\gamma}^\alpha \Theta^\beta \wedge \Theta^\gamma \\ &= \left( B_{b\gamma}^\alpha \Theta^b + \frac{1}{2} C_{\beta\gamma}^\alpha \Theta^\beta \right) \wedge \Theta^\gamma. \end{aligned}$$

As

$$\Omega_\gamma^\alpha \stackrel{put}{=} B_{b\gamma}^\alpha \Theta^b + \frac{1}{2} C_{\beta\gamma}^\alpha \Theta^\beta \in \Lambda^1(F, \nu, N),$$

for any  $\alpha, \beta \in \overline{r+1, p}$ , it results the first implication.

Conversely, we admit that it exists

$$\Omega_\beta^\alpha \in \Lambda^1(F, \nu, N), \quad \alpha, \beta \in \overline{r+1, p}$$

such that

$$(4) \quad d^F \Theta^\alpha = \sum_{\beta \in \overline{r+1, p}} \Omega_\beta^\alpha \wedge \Theta^\beta,$$

for any  $\alpha \in \overline{r+1, p}$ .

Using the affirmations (1), (2) and (4) we obtain that  $A_{bc}^\alpha = 0$ , for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ .

Using the affirmation (3), we obtain  $\Theta^\alpha([S_b, S_c]_F) = 0$ , for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ .

Therefore, we have  $[S_b, S_c]_F \in \Gamma(E, \pi, N)$ , for any  $b, c \in \overline{1, r}$ . Using the *Proposition 3.2.2*, we obtain the second implication. *q.e.d.*

## 5 Exterior Differential Systems

Let  $((F, \nu, N), [, ]_F, (\rho, Id_N))$  be a Lie algebroid.

**Definition 5.1** Any ideal  $(\mathcal{I}, +, \cdot)$  of the exterior differential algebra of the Lie algebroid  $((F, \nu, N), [, ]_F, (\rho, Id_M))$  closed under differentiation operator  $d^F$ , namely  $d^F \mathcal{I} \subseteq \mathcal{I}$ , is called *differential ideal of the Lie algebroid*  $((F, \nu, N), [, ]_F, (\rho, Id_M))$ .

**Definition 5.2** Let  $(\mathcal{I}, +, \cdot)$  be a differential ideal of the Lie algebroid

$$((F, \nu, N), [\cdot]_F, (\rho, Id_M)).$$

If it exists an IDS  $(E, \pi, N)$  such that for all  $k \in \mathbb{N}^*$  and  $\omega \in \mathcal{I} \cap \Lambda^k(F, \nu, N)$  we have  $\omega(u_1, \dots, u_k) = 0$ , for any  $u_1, \dots, u_k \in \Gamma(E, \pi, N)$ , then we will say that  $(\mathcal{I}, +, \cdot)$  is an exterior differential system (EDS) of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$ .

**Theorem 5.1** (of Cartan type) *The IDS  $(E, \pi, N)$  of the Lie algebroid*

$$((F, \nu, N), [\cdot]_F, (\rho, Id_N))$$

*is involutive, if and only if the ideal generated by the  $\mathcal{F}(N)$ -submodule  $(\Gamma(E^0, \pi^0, N), +, \cdot)$  is an EDS of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$ .*

*Proof.* Let  $(E, \pi, N)$  be an involutive IDS of the Lie algebroid

$$((F, \nu, N), [\cdot]_F, (\rho, Id_N)).$$

Let  $\{\Theta^{r+1}, \dots, \Theta^p\}$  be a base for the  $\mathcal{F}(N)$ -submodule  $(\Gamma(E^0, \pi^0, N), +, \cdot)$ .

We know that

$$\mathcal{I}(\Gamma(E^0, \pi^0, N)) = \cup_{q \in \mathbb{N}} \{\Omega_\alpha \wedge \Theta^\alpha, \{\Omega_{r+1}, \dots, \Omega_p\} \subset \Lambda^q(F, \nu, N)\}.$$

Let  $q \in \mathbb{N}$  and  $\{\Omega_{r+1}, \dots, \Omega_p\} \subset \Lambda^q(F, \nu, N)$  be arbitrary.

Using the *Theorems 4.5 and 4.7* we obtain

$$\begin{aligned} d^F(\Omega_\alpha \wedge \Theta^\alpha) &= d^F \Omega_\alpha \wedge \Theta^\alpha + (-1)^{q+1} \Omega_\beta \wedge d^F \Theta^\beta \\ &= (d^F \Omega_\alpha + (-1)^{q+1} \Omega_\beta \wedge \Omega_\alpha^\beta) \wedge \Theta^\alpha. \end{aligned}$$

As

$$d^F \Omega_\alpha + (-1)^{q+1} \Omega_\beta \wedge \Omega_\alpha^\beta \in \Lambda^{q+2}(F, \nu, N)$$

it results that

$$d^F(\Omega_\beta \wedge \Theta^\beta) \in \mathcal{I}(\Gamma(E^0, \pi^0, N))$$

Therefore,

$$d^F \mathcal{I}(\Gamma(E^0, \pi^0, N)) \subseteq \mathcal{I}(\Gamma(E^0, \pi^0, N)).$$

Conversely, let  $(E, \pi, N)$  be an IDS of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$  such that the  $\mathcal{F}(N)$ -submodule  $(\mathcal{I}(\Gamma(E^0, \pi^0, N)), +, \cdot)$  is an EDS of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$ .

Let  $\{\Theta^{r+1}, \dots, \Theta^p\}$  be a base for the  $\mathcal{F}(N)$ -submodule  $(\Gamma(E^0, \pi^0, N), +, \cdot)$ . As

$$d^F \mathcal{I}(\Gamma(E^0, \pi^0, N)) \subseteq \mathcal{I}(\Gamma(E^0, \pi^0, N))$$

it results that it exists

$$\Omega_\beta^\alpha \in \Lambda^1(F, \nu, N), \quad \alpha, \beta \in \overline{r+1, p}$$

such that

$$d^F \Theta^\alpha = \sum_{\beta \in \overline{r+1, p}} \Omega_\beta^\alpha \wedge \Theta^\beta \in \mathcal{I}(\Gamma(E^0, \pi^0, N)).$$

Using the *Theorem 4.7*, it results that  $(E, \pi, N)$  is an involutive IDS.

*q.e.d.*



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# INTERIOR AND EXTERIOR DIFFERENTIAL SYSTEMS FOR LIE ALGEBROIDS

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*In memory of my uncle  
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## Abstract

A theorem of Maurer-Cartan type for Lie algebroids is presented. Suppose that any vector subbundle of a Lie algebroid is called *interior differential system (IDS)* for that Lie algebroid. A theorem of Cartan type is obtained. Extending the classical notion of *exterior differential system (EDS)* to Lie algebroids, a theorem of Cartan type is obtained.

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## 1 Introduction

Using the exterior differential calculus for Lie algebroids (see: [2, 6]) the structure equations of Maurer-Cartan type are established.

Using the *Cartan's moving frame method*, there exists the following

**Theorem** (E. Cartan) *If  $N \in |\mathbf{Man}_n|$  is a Riemannian manifold and  $X_\alpha = X_\alpha^i \frac{\partial}{\partial x^i}$ ,  $\alpha \in \overline{1, n}$  is an orthonormal moving frame, then there exists a collection of 1-forms  $\Omega_\beta^\alpha$ ,  $\alpha, \beta \in \overline{1, n}$  uniquely defined by the requirements*

$$\Omega_\beta^\alpha = -\Omega_\alpha^\beta$$

and

$$d^F \Theta^\alpha = \Omega_\beta^\alpha \wedge \Theta^\beta, \quad \alpha \in \overline{1, n}$$

where  $\{\Theta^\alpha, \alpha \in \overline{1, n}\}$  is the coframe. (see [5], p. 151)

We know that an *r-dimensional distribution on a manifold  $N$*  is a mapping  $\mathcal{D}$  defined on  $N$ , which assigns to each point  $x$  of  $N$  an  $r$ -dimensional linear subspace  $\mathcal{D}_x$  of  $T_x N$ . A vector fields  $X$  belongs to  $\mathcal{D}$  if we have  $X_x \in \mathcal{D}_x$  for each  $x \in N$ . When this happens we write  $X \in \Gamma(\mathcal{D})$ .

The distribution  $\mathcal{D}$  on a manifold  $N$  is said to be *differentiable* if for any  $x \in N$  there exists  $r$  differentiable linearly independent vector fields  $X_1, \dots, X_r \in \Gamma(\mathcal{D})$  in a

neighborhood of  $x$ . The distribution  $\mathcal{D}$  is said to be *involutive* if for all vector fields  $X, Y \in \Gamma(\mathcal{D})$  we have  $[X, Y] \in \Gamma(\mathcal{D})$ .

Extending the notion of distribution we obtain the definition of an *IDS* of a Lie algebroid. A characterization of the involutivity of an *IDS* in a result of Cartan type is presented in *Theorem 4.7*.

In the classical theory we have the following

**Theorem** (Frobenius) *The distribution  $\mathcal{D}$  is involutive if and only if for each  $x \in N$  there exists a neighborhood  $U$  and  $n - r$  linearly independent 1-forms  $\Theta^{r+1}, \dots, \Theta^n$  on  $U$  which vanish on  $\mathcal{D}$  and satisfy the condition*

$$d^F \Theta^\alpha = \sum_{\beta \in \overline{r+1, n}} \Omega_\beta^\alpha \wedge \Theta^\beta, \quad \alpha \in \overline{r+1, n}.$$

for suitable 1-forms  $\Omega_\beta^\alpha$ ,  $\alpha, \beta \in \overline{r+1, n}$ . (see [4], p. 58)

This paper studies the intersection between the geometry of Lie algebroids and some aspects of *EDS*. In the classical sense, an *EDS* is a pair  $(M, \mathcal{E})$  consisting of a smooth manifold  $M$  and a homogeneous, differentially closed ideal  $\mathcal{E} \subseteq \Omega^*(M)$  in the algebra of smooth differential forms on  $M$ . ( see [1, 3]) Using the notion of *EDS* of an arbitrary Lie algebroid  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$  we obtained a new result of Cartan type in the *Theorem 5.1*.

In the particular case of standard Lie algebroid  $((TM, \tau_M, M), [\cdot, \cdot]_{TM}, (Id_{TM}, Id_M))$  there are obtained similar results those for distributions.

We know that a submanifold  $S$  of  $N$  is said to be *integral manifold* for the distribution  $\mathcal{D}$  if for every point  $x \in N$ ,  $\mathcal{D}_x$  coincides with  $T_x S$ . The distribution  $\mathcal{D}$  is said to be *integrable* if for each point  $x \in N$  there exists an integral manifold of  $\mathcal{D}$  containing  $x$ .

As a distribution  $\mathcal{D}$  is involutive if and only if it is integrable, then the study of the integral manifolds of an *IDS* or *EDS* is a new direction by research.

## 2 Preliminaries

In general, if  $\mathcal{C}$  is a category, then we denote  $|\mathcal{C}|$  the class of objects and for any  $A, B \in |\mathcal{C}|$ , we denote  $\mathcal{C}(A, B)$  the set of morphisms of  $A$  source and  $B$  target. Let **Liealg**, **Mod**, and **B<sup>v</sup>** be the category of Lie algebras, modules and vector bundles respectively.

We know that if  $(E, \pi, M) \in |\mathbf{B}^v|$ ,  $\Gamma(E, \pi, M) = \{u \in \mathbf{Man}(M, E) : u \circ \pi = Id_M\}$  and  $\mathcal{F}(M) = \mathbf{Man}(M, \mathbb{R})$ , then  $(\Gamma(E, \pi, M), +, \cdot)$  is a  $\mathcal{F}(M)$ -module. Additionally, if  $(E, \pi, M) \in |\mathbf{B}^v|$  so that  $M$  is paracompact and if  $A \subseteq M$  is closed, then for any section  $u$  over  $A$  it exists  $\tilde{u} \in \Gamma(E, \pi, M)$  so that  $\tilde{u}|_A = u$ . In the following, we consider only vector bundles with paracompact base.

We know that a Lie algebroid is a vector bundle  $(F, \nu, N) \in |\mathbf{B}^v|$  so that there exists

$$(\rho, Id_N) \in \mathbf{B}^v((F, \nu, N), (TN, \tau_N, N))$$

and an operation

$$\begin{array}{ccc} \Gamma(F, \nu, N) \times \Gamma(F, \nu, N) & \xrightarrow{[\cdot]_F} & \Gamma(F, \nu, N) \\ (u, v) & \longmapsto & [u, v]_F \end{array}$$

with the following properties:

$LA_1$ . the equality holds good

$$[u, f \cdot v]_F = f[u, v]_F + \Gamma(\rho, Id_N)(u) f \cdot v,$$

for all  $u, v \in \Gamma(F, \nu, N)$  and  $f \in \mathcal{F}(N)$ ,

$LA_2$ . the 4-tuple  $(\Gamma(F, \nu, N), +, \cdot, [, ]_F)$  is a Lie  $\mathcal{F}(N)$ -algebra,

$LA_3$ . the **Mod**-morphism  $\Gamma(\rho, Id_N)$  is a **LieAlg**-morphism of  $(\Gamma(F, \nu, N), +, \cdot, [, ]_F)$  source and  $(\Gamma(TN, \tau_N, N), +, \cdot, [, ]_{TN})$  target.

Let  $\left((F, \nu, N), [, ]_F, (\rho, Id_N)\right)$  be a Lie algebroid.

- Locally, for any  $\alpha, \beta \in \overline{1, p}$ , we set  $[t_\alpha, t_\beta]_F = L_{\alpha\beta}^\gamma t_\gamma$ . We easily obtain that  $L_{\alpha\beta}^\gamma = -L_{\beta\alpha}^\gamma$ , for any  $\alpha, \beta, \gamma \in \overline{1, p}$ .

The real local functions  $\left\{L_{\alpha\beta}^\gamma, \alpha, \beta, \gamma \in \overline{1, p}\right\}$  are called the *structure functions*.

- We assume that  $(F, \nu, N)$  is a vector bundle with type fibre the real vector space  $(\mathbb{R}^p, +, \cdot)$  and structure group a Lie subgroup of  $(\mathbf{GL}(p, \mathbb{R}), \cdot)$ . We denote  $(x^i, z^\alpha)$  the canonical local coordinates on  $(F, \nu, N)$ , where  $i \in \overline{1, n}$ ,  $\alpha \in \overline{1, p}$ .

Consider

$$(x^i, z^\alpha) \longrightarrow (x^{i'}, z^{\alpha'})$$

a change of coordinates on  $(F, \nu, N)$ . Then the coordinates  $z^\alpha$  change to  $z^{\alpha'}$  according to the rule:

$$(2.1) \quad z^{\alpha'} = \Lambda_{\alpha}^{\alpha'} z^\alpha.$$

- If  $z^\alpha t_\alpha \in \Gamma(F, \nu, N)$  is arbitrary, then

$$(2.2) \quad [\Gamma(\rho, Id_N)(z^\alpha t_\alpha) f](x) = \left(\rho_\alpha^i z^\alpha \frac{\partial f}{\partial x^i}\right)(x)$$

for any  $f \in \mathcal{F}(N)$  and  $x \in N$ .

The coefficients  $\rho_\alpha^i$  change to  $\rho_{\alpha'}^{i'}$  according to the rule:

$$(2.3) \quad \rho_{\alpha'}^{i'} = \Lambda_{\alpha}^{\alpha'} \rho_\alpha^i \frac{\partial x^{i'}}{\partial x^i},$$

where

$$\|\Lambda_{\alpha}^{\alpha'}\| = \left\|\Lambda_{\alpha}^{\alpha'}\right\|^{-1}.$$

The following equalities hold good:

$$(2.4) \quad \left(\rho_\alpha^i \frac{\partial}{\partial x^i}\right)(f) = \rho_\alpha^i \frac{\partial f}{\partial x^i}, \forall f \in \mathcal{F}(N).$$

and

$$(2.5) \quad L_{\alpha\beta}^\gamma \cdot \rho_\gamma^k = \rho_\alpha^i \cdot \frac{\partial \rho_\beta^k}{\partial x^i} - \rho_\beta^j \cdot \frac{\partial \rho_\alpha^k}{\partial x^j}.$$

### 3 Interior Differential Systems

Let  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$  be a Lie algebroid.

**Definition 3.1** Any vector subbundle  $(E, \pi, N)$  of the vector bundle  $(F, \nu, N)$  will be called *interior differential system (IDS) of the Lie algebroid*

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N)).$$

*Remark 3.1* If  $(E, \pi, M)$  is an IDS of the Lie algebroid

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N)),$$

then we obtain a vector subbundle  $(E^0, \pi^0, N)$  of the dual vector bundle  $(F^*, \nu^*, N)$  so that

$$\Gamma(E^0, \pi^0, N) \stackrel{put}{=} \left\{ \Omega \in \Gamma(F^*, \nu^*, N) : \Omega(S) = 0, \forall S \in \Gamma(E, \pi, N) \right\}.$$

The vector subbundle  $(E^0, \pi^0, N)$  will be called *the annihilator vector subbundle of the IDS  $(E, \pi, N)$* .

**Proposition 3.1** If  $(E, \pi, N)$  is an IDS of the Lie algebroid

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$$

so that  $\Gamma(E, \pi, N) = \langle S_1, \dots, S_r \rangle$ , then it exists  $\Theta^{r+1}, \dots, \Theta^p \in \Gamma(F^*, \nu^*, N)$  linearly independent so that  $\Gamma(E^0, \pi^0, N) = \langle \Theta^{r+1}, \dots, \Theta^p \rangle$ .

**Definition 3.2** The IDS  $(E, \pi, N)$  of the Lie algebroid

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$$

will be called *involutive* if  $[S, T]_F \in \Gamma(E, \pi, N)$ , for any  $S, T \in \Gamma(E, \pi, N)$ .

**Proposition 3.2** If  $(E, \pi, N)$  is an IDS of the Lie algebroid

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, \eta))$$

and  $\{S_1, \dots, S_r\}$  is a base of the  $\mathcal{F}(M)$ -submodule  $(\Gamma(E, \pi, N), +, \cdot)$  then  $(E, \pi, N)$  is involutive if and only if  $[S_a, S_b]_F \in \Gamma(E, \pi, N)$ , for any  $a, b \in \overline{1, r}$ .

### 4 Exterior differential calculus

Let  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$  be a Lie algebroid.

We denote  $\Lambda^q(F, \nu, N)$  the set of *differential forms of degree  $q$* . If

$$\Lambda(F, \nu, N) = \bigoplus_{q \geq 0} \Lambda^q(F, \nu, N),$$

then we obtain *the exterior differential algebra*  $(\Lambda(F, \nu, N), +, \cdot, \wedge)$ .

**Definition 4.1** For any  $z \in \Gamma(F, \nu, N)$ , the application

$$\Lambda(F, \nu, N) \xrightarrow{L_z} \Lambda(F, \nu, N),$$

defined by

$$L_z(f) = [\Gamma(\rho, Id_N)z](f),$$

for any  $f \in \mathcal{F}(N)$  and

$$\begin{aligned} L_z\omega(z_1, \dots, z_q) &= [\Gamma(\rho, Id_N)z](\omega((z_1, \dots, z_q))) \\ &\quad - \sum_{i=1}^q \omega((z_1, \dots, [z, z_i]_F, \dots, z_q)), \end{aligned}$$

for any  $\omega \in \Lambda^q(F, \nu, N)$  and  $z_1, \dots, z_q \in \Gamma(F, \nu, N)$ , is called *the covariant Lie derivative with respect to the section  $z$* .

**Theorem 4.1** *If  $z \in \Gamma(F, \nu, N)$ ,  $\omega \in \Lambda^q(F, \nu, N)$  and  $\theta \in \Lambda^r(F, \nu, N)$ , then*

$$(4.1) \quad L_z(\omega \wedge \theta) = L_z\omega \wedge \theta + \omega \wedge L_z\theta.$$

**Definition 4.2** For any  $z \in \Gamma(F, \nu, N)$ , the application

$$\begin{aligned} \Lambda(F, \nu, N) &\xrightarrow{i_z} \Lambda(F, \nu, N) \\ \Lambda^q(F, \nu, N) \ni \omega &\longmapsto i_z\omega \in \Lambda^{q-1}(F, \nu, N), \end{aligned}$$

defined by  $i_z f = 0$ , for any  $f \in \mathcal{F}(N)$  and

$$i_z\omega(z_2, \dots, z_q) = \omega(z, z_2, \dots, z_q),$$

for any  $z_2, \dots, z_q \in \Gamma(F, \nu, N)$ , is called *the interior product associated to the section  $z$* .

**Theorem 4.2** *If  $z \in \Gamma(F, \nu, N)$ , then for any  $\omega \in \Lambda^q(F, \nu, N)$  and  $\theta \in \Lambda^r(F, \nu, N)$  we obtain*

$$(4.2) \quad i_z(\omega \wedge \theta) = i_z\omega \wedge \theta + (-1)^q \omega \wedge i_z\theta.$$

**Theorem 4.3** *For any  $z, v \in \Gamma(F, \nu, N)$  we obtain*

$$(4.3) \quad L_v \circ i_z - i_z \circ L_v = i_{[z, v]_F}.$$

**Theorem 4.4** *The application*

$$\begin{aligned} \Lambda^q(F, \nu, N) &\xrightarrow{d^F} \Lambda^{q+1}(F, \nu, N) \\ \omega &\longmapsto d\omega \end{aligned}$$

defined by

$$d^F f(z) = \Gamma(\rho, Id_N)(z)f,$$

for any  $z \in \Gamma(F, \nu, N)$ , and

$$\begin{aligned} d^F\omega(z_0, z_1, \dots, z_q) &= \sum_{i=0}^q (-1)^i \Gamma(\rho, Id_N)z_i(\omega((z_0, z_1, \dots, \hat{z}_i, \dots, z_q))) \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([z_i, z_j]_F, z_0, z_1, \dots, \hat{z}_i, \dots, \hat{z}_j, \dots, z_q), \end{aligned}$$

for any  $z_0, z_1, \dots, z_q \in \Gamma(F, \nu, N)$ , is unique having the following property:

$$(4.4) \quad L_z = d^F \circ i_z + i_z \circ d^F, \quad \forall z \in \Gamma(F, \nu, N).$$

This application is called *the exterior differentiation operator of the exterior differential algebra of the Lie algebroid*  $((F, \nu, N), [, ]_F, (\rho, Id_N))$ .

**Theorem 4.5** *The exterior differentiation operator  $d^F$  given by the previous theorem has the following properties:*

1. *For any  $\omega \in \Lambda^q(F, \nu, N)$  and  $\theta \in \Lambda^r(F, \nu, N)$  we obtain*

$$(4.5) \quad d^F(\omega \wedge \theta) = d^F\omega \wedge \theta + (-1)^q \omega \wedge d^F\theta.$$

2. *For any  $z \in \Gamma(F, \nu, N)$  we obtain*

$$(4.6) \quad L_z \circ d^F = d^F \circ L_z.$$

3.  $d^F \circ d^F = 0$ .

**Theorem 4.6** (of Maurer-Cartan type)

*If  $((F, \nu, N), [, ]_F, (\rho, Id_N))$  is a Lie algebroid and  $d^F$  is the exterior differentiation operator of the exterior differential  $\mathcal{F}(N)$ -algebra  $(\Lambda(F, \nu, N), +, \cdot, \wedge)$ , then we obtain the structure equations of Maurer-Cartan type*

$$(C_1) \quad d^F t^\alpha = -\frac{1}{2} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma, \quad \alpha \in \overline{1, p}$$

and

$$(C_2) \quad d^F x^i = \rho_\alpha^i t^\alpha, \quad i \in \overline{1, n},$$

where  $\{t^\alpha, \alpha \in \overline{1, p}\}$  is the coframe of the vector bundle  $(F, \nu, N)$ .

This equations will be called *the structure equations of Maurer-Cartan type associated to the Lie algebroid*  $((F, \nu, N), [, ]_F, (\rho, Id_N))$ .

*Proof.* Let  $\alpha \in \overline{1, p}$  be arbitrary. Since

$$d^F t^\alpha(t_\beta, t_\gamma) = -L_{\beta\gamma}^\alpha, \quad \forall \beta, \gamma \in \overline{1, p}$$

it results that

$$(1) \quad d^F t^\alpha = -\sum_{\beta < \gamma} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma.$$

Since  $L_{\beta\gamma}^\alpha = -L_{\gamma\beta}^\alpha$  and  $t^\beta \wedge t^\gamma = -t^\gamma \wedge t^\beta$ , for nay  $\beta, \gamma \in \overline{1, p}$ , it results that

$$(2) \quad \sum_{\beta < \gamma} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma = \frac{1}{2} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma$$

Using the equalities (1) and (2) it results the structure equation  $(C_1)$ .

Let  $i \in \overline{1, n}$  be arbitrary. Since

$$d^F x^i(t_\alpha) = \rho_\alpha^i, \quad \forall \alpha \in \overline{1, p}$$

it results the structure equation  $(C_2)$ .

*q.e.d.*

**Remark 4.1** In the particular case of the standard Lie algebroid

$$((TN, \tau_N, N), [\cdot, \cdot]_{TN}, (Id_{TN}, Id_N))$$

we obtain

$$(\mathcal{C}'_2) \quad d^{TN} x^i = dx^i, \quad i \in \overline{1, n},$$

where  $\{dx^i, i \in \overline{1, n}\}$  is the coframe of the vector bundle  $(TN, \tau_N, N)$ .

As  $d^{TN} \circ d^{TN} = 0$  and  $L_{jk}^i = 0$ , for all  $i, j, k \in \overline{1, n}$  we obtain

$$(\mathcal{C}'_1) \quad d^F(dx^i) = 0 = -\frac{1}{2}L_{jk}^i dx^j \wedge dx^k, \quad i \in \overline{1, n}$$

This equations are the structure equations of Maurer-Cartan type associated to the standard Lie algebroid  $((TN, \tau_N, N), [\cdot, \cdot]_{TN}, (Id_{TN}, Id_N))$ .

**Theorem 4.7** (of Cartan type) *Let  $(E, \pi, N)$  be an IDS of the Lie algebroid*

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N)).$$

*If  $\{\Theta^{r+1}, \dots, \Theta^p\}$  is a base of the  $\mathcal{F}(N)$ -submodule  $(\Gamma(E^0, \pi^0, N), +, \cdot)$ , then the IDS  $(E, \pi, N)$  is involutive if and only if it exists*

$$\Omega_\beta^\alpha \in \Lambda^1(F, \nu, N), \quad \alpha, \beta \in \overline{r+1, p}$$

*so that*

$$d^F \Theta^\alpha = \sum_{\beta \in \overline{r+1, p}} \Omega_\beta^\alpha \wedge \Theta^\beta \in \mathcal{I}(\Gamma(E^0, \pi^0, N)).$$

*Proof.* Let  $\{S_1, \dots, S_r\}$  be a base of the  $\mathcal{F}(N)$ -submodule  $(\Gamma(E, \pi, N), +, \cdot)$

Let  $\{S_{r+1}, \dots, S_p\} \in \Gamma(F, \nu, N)$  so that  $\{S_1, \dots, S_r, S_{r+1}, \dots, S_p\}$  is a base of the  $\mathcal{F}(N)$ -module

$$(\Gamma(F, \nu, N), +, \cdot).$$

Let  $\Theta^1, \dots, \Theta^r \in \Gamma\left(\overset{*}{F}, \overset{*}{\nu}, N\right)$  so that  $\{\Theta^1, \dots, \Theta^r, \Theta^{r+1}, \dots, \Theta^p\}$  is a base of the  $\mathcal{F}(N)$ -module

$$\left(\Gamma\left(\overset{*}{F}, \overset{*}{\nu}, N\right), +, \cdot\right).$$

For any  $a, b \in \overline{1, r}$  and  $\alpha, \beta \in \overline{r+1, p}$ , we have the equalities:

$$\begin{aligned} \Theta^a(S_b) &= \delta_b^a \\ \Theta^a(S_\beta) &= 0 \\ \Theta^\alpha(S_b) &= 0 \\ \Theta^\alpha(S_\beta) &= \delta_\beta^\alpha \end{aligned}$$

We remark that the set of the 2-forms

$$\left\{ \Theta^a \wedge \Theta^b, \Theta^a \wedge \Theta^\beta, \Theta^\alpha \wedge \Theta^\beta, \quad a, b \in \overline{1, r} \wedge \alpha, \beta \in \overline{r+1, p} \right\}$$

is a base of the  $\mathcal{F}(M)$ -module  $(\Lambda^2(F, \nu, N), +, \cdot)$ .



Therefore, we have

$$(1) \quad d^F \Theta^\alpha = \Sigma_{b < c} A_{bc}^\alpha \Theta^b \wedge \Theta^c + \Sigma_{b, \gamma} B_{b\gamma}^\alpha \Theta^b \wedge \Theta^\gamma + \Sigma_{\beta < \gamma} C_{\beta\gamma}^\alpha \Theta^\beta \wedge \Theta^\gamma,$$

where,  $A_{bc}^\alpha, B_{b\gamma}^\alpha$  and  $C_{\beta\gamma}^\alpha$ ,  $a, b, c \in \overline{1, r}$ ,  $\alpha, \beta, \gamma \in \overline{r+1, p}$  are real local functions so that  $A_{bc}^\alpha = -A_{cb}^\alpha$  and  $C_{\beta\gamma}^\alpha = -C_{\gamma\beta}^\alpha$ .

Using the formula

$$(2) \quad \begin{aligned} d^F \Theta^\alpha (S_b, S_c) &= \Gamma(\rho, Id_N) S_b (\Theta^\alpha (S_c)) - \Gamma(\rho, Id_N) S_c (\Theta^\alpha (S_b)) \\ &\quad - \Theta^\alpha ([S_b, S_c]_F), \end{aligned}$$

we obtain that

$$(3) \quad A_{bc}^\alpha = -\Theta^\alpha ([S_b, S_c]_F),$$

for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ .

We admit that  $(E, \pi, N)$  is an involutive *IDS* of the Lie algebroid  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$ .

As  $[S_b, S_c]_F \in \Gamma(E, \pi, N)$ , for any  $b, c \in \overline{1, r}$ , it results that  $\Theta^\alpha ([S_b, S_c]_F) = 0$ , for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ . Therefore, for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ , we obtain  $A_{bc}^\alpha = 0$  and

$$\begin{aligned} d^F \Theta^\alpha &= \Sigma_{b, \gamma} B_{b\gamma}^\alpha \Theta^b \wedge \Theta^\gamma + \frac{1}{2} C_{\beta\gamma}^\alpha \Theta^\beta \wedge \Theta^\gamma \\ &= \left( B_{b\gamma}^\alpha \Theta^b + \frac{1}{2} C_{\beta\gamma}^\alpha \Theta^\beta \right) \wedge \Theta^\gamma. \end{aligned}$$

As

$$\Omega_\gamma^\alpha \stackrel{put}{=} B_{b\gamma}^\alpha \Theta^b + \frac{1}{2} C_{\beta\gamma}^\alpha \Theta^\beta \in \Lambda^1(F, \nu, N),$$

for any  $\alpha, \beta \in \overline{r+1, p}$ , it results the first implication.

Conversely, we admit that it exists

$$\Omega_\beta^\alpha \in \Lambda^1(F, \nu, N), \quad \alpha, \beta \in \overline{r+1, p}$$

so that

$$(4) \quad d^F \Theta^\alpha = \Sigma_{\beta \in \overline{r+1, p}} \Omega_\beta^\alpha \wedge \Theta^\beta,$$

for any  $\alpha \in \overline{r+1, p}$ .

Using the affirmations (1), (2) and (4) we obtain that  $A_{bc}^\alpha = 0$ , for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ .

Using the affirmation (3), we obtain  $\Theta^\alpha ([S_b, S_c]_F) = 0$ , for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ .

Therefore, we have  $[S_b, S_c]_F \in \Gamma(E, \pi, N)$ , for any  $b, c \in \overline{1, r}$ . Using the *Proposition 3.2.2*, we obtain the second implication. *q.e.d.*

## 5 Exterior Differential Systems

Let  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$  be a Lie algebroid.

**Definition 5.1** Any ideal  $(\mathcal{I}, +, \cdot)$  of the exterior differential algebra of the Lie algebroid  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_M))$  closed under differentiation operator  $d^F$ , namely  $d^F \mathcal{I} \subseteq \mathcal{I}$ , is called *differential ideal of the Lie algebroid*  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_M))$ .

**Definition 5.2** Let  $(\mathcal{I}, +, \cdot)$  be a differential ideal of the Lie algebroid

$$((F, \nu, N), [\cdot]_F, (\rho, Id_M)).$$

If it exists an *IDS*  $(E, \pi, N)$  so that for all  $k \in \mathbb{N}^*$  and  $\omega \in \mathcal{I} \cap \Lambda^k(F, \nu, N)$  we have  $\omega(u_1, \dots, u_k) = 0$ , for any  $u_1, \dots, u_k \in \Gamma(E, \pi, N)$ , then we will say that  $(\mathcal{I}, +, \cdot)$  is an *exterior differential system (EDS)* of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$ .

**Theorem 5.1** (of Cartan type) *The IDS  $(E, \pi, N)$  of the Lie algebroid*

$$((F, \nu, N), [\cdot]_F, (\rho, Id_N))$$

*is involutive, if and only if the ideal generated by the  $\mathcal{F}(N)$ -submodule  $(\Gamma(E^0, \pi^0, N), +, \cdot)$  is an EDS of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$ .*

*Proof.* Let  $(E, \pi, N)$  be an involutive *IDS* of the Lie algebroid

$$((F, \nu, N), [\cdot]_F, (\rho, Id_N)).$$

Let  $\{\Theta^{r+1}, \dots, \Theta^p\}$  be a base of the  $\mathcal{F}(N)$ -submodule  $(\Gamma(E^0, \pi^0, N), +, \cdot)$ .

We know that

$$\mathcal{I}(\Gamma(E^0, \pi^0, N)) = \cup_{q \in \mathbb{N}} \{\Omega_\alpha \wedge \Theta^\alpha, \{\Omega_{r+1}, \dots, \Omega_p\} \subset \Lambda^q(F, \nu, N)\}.$$

Let  $q \in \mathbb{N}$  and  $\{\Omega_{r+1}, \dots, \Omega_p\} \subset \Lambda^q(F, \nu, N)$  be arbitrary.

Using the *Theorems 4.5 and 4.7* we obtain

$$\begin{aligned} d^F(\Omega_\alpha \wedge \Theta^\alpha) &= d^F \Omega_\alpha \wedge \Theta^\alpha + (-1)^{q+1} \Omega_\beta \wedge d^F \Theta^\beta \\ &= \left( d^F \Omega_\alpha + (-1)^{q+1} \Omega_\beta \wedge \Omega_\alpha^\beta \right) \wedge \Theta^\alpha. \end{aligned}$$

As

$$d^F \Omega_\alpha + (-1)^{q+1} \Omega_\beta \wedge \Omega_\alpha^\beta \in \Lambda^{q+2}(F, \nu, N)$$

it results that

$$d^F(\Omega_\beta \wedge \Theta^\beta) \in \mathcal{I}(\Gamma(E^0, \pi^0, N))$$

Therefore,

$$d^F \mathcal{I}(\Gamma(E^0, \pi^0, N)) \subseteq \mathcal{I}(\Gamma(E^0, \pi^0, N)).$$

Conversely, let  $(E, \pi, N)$  be an *IDS* of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$  so that the  $\mathcal{F}(N)$ -submodule  $(\mathcal{I}(\Gamma(E^0, \pi^0, N)), +, \cdot)$  is an *EDS* of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$ .

Let  $\{\Theta^{r+1}, \dots, \Theta^p\}$  be a base of the  $\mathcal{F}(N)$ -submodule  $(\Gamma(E^0, \pi^0, N), +, \cdot)$ . As

$$d^F \mathcal{I}(\Gamma(E^0, \pi^0, N)) \subseteq \mathcal{I}(\Gamma(E^0, \pi^0, N))$$

it results that it exists

$$\Omega_\beta^\alpha \in \Lambda^1(F, \nu, N), \quad \alpha, \beta \in \overline{r+1, p}$$

so that

$$d^F \Theta^\alpha = \sum_{\beta \in \overline{r+1, p}} \Omega_\beta^\alpha \wedge \Theta^\beta \in \mathcal{I}(\Gamma(E^0, \pi^0, N)).$$

Using the *Theorem 4.7* there results that  $(E, \pi, N)$  is an involutive *IDS*. *q.e.d.*

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# INTERIOR AND EXTERIOR DIFFERENTIAL SYSTEMS FOR LIE ALGEBROIDS

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*In memory of  
Prof. Dr. Gheorghe RADU*

*Dedicated to  
Acad. Prof. Dr. Doc. Radu MIRON at his 84<sup>th</sup> anniversary*

## Abstract

A theorem of Maurer-Cartan type for Lie algebroids is presented. Suppose that any vector subbundle of a Lie algebroid is called *interior differential system (IDS)* for that Lie algebroid. A theorem of Frobenius type is obtained. Extending the classical notion of *exterior differential system (EDS)* to Lie algebroids, a theorem of Cartan type is obtained.

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## 1 Introduction

Using the exterior differential calculus for Lie algebroids (see: [1, 2]) the structure equations of Maurer-Cartan type are established. Using the *Cartan's moving frame method*, there exists the following

**Theorem** (E. Cartan) *If  $N \in |\mathbf{Man}_n|$  is a Riemannian manifold and  $X_\alpha = X_\alpha^i \frac{\partial}{\partial x^i}$ ,  $\alpha \in \overline{1, n}$  is an orthonormal moving frame, then there exists a collection of 1-forms  $\Omega_\beta^\alpha$ ,  $\alpha, \beta \in \overline{1, n}$  uniquely defined by the requirements*

$$\Omega_\beta^\alpha = -\Omega_\alpha^\beta$$

and

$$d^F \Theta^\alpha = \Omega_\beta^\alpha \wedge \Theta^\beta, \quad \alpha \in \overline{1, n}$$

where  $\{\Theta^\alpha, \alpha \in \overline{1, n}\}$  is the coframe. (see [3], p. 151)

We know that an  $r$ -dimensional distribution on a manifold  $N$  is a mapping  $\mathcal{D}$  defined on  $N$ , which assigns to each point  $x$  of  $N$  an  $r$ -dimensional linear subspace  $\mathcal{D}_x$  of  $T_x N$ .

A vector fields  $X$  belongs to  $\mathcal{D}$  if we have  $X_x \in \mathcal{D}_x$  for each  $x \in N$ . When this happens we write  $X \in \Gamma(\mathcal{D})$ .

The distribution  $\mathcal{D}$  on a manifold  $N$  is said to be *differentiable* if for any  $x \in N$  there exists  $r$  differentiable linearly independent vector fields  $X_1, \dots, X_r \in \Gamma(\mathcal{D})$  in a neighborhood of  $x$ . The distribution  $\mathcal{D}$  is said to be *involutive* if for all vector fields  $X, Y \in \Gamma(\mathcal{D})$  we have  $[X, Y] \in \Gamma(\mathcal{D})$ .

In the classical theory we have the following

**Theorem** (Frobenius) *The distribution  $\mathcal{D}$  is involutive if and only if for each  $x \in N$  there exists a neighborhood  $U$  and  $n - r$  linearly independent 1-forms  $\Theta^{r+1}, \dots, \Theta^n$  on  $U$  which vanish on  $\mathcal{D}$  and satisfy the condition*

$$d^F \Theta^\alpha = \sum_{\beta \in \overline{r+1, p}} \Omega_\beta^\alpha \wedge \Theta^\beta, \quad \alpha \in \overline{r+1, n}.$$

for suitable 1-forms  $\Omega_\beta^\alpha$ ,  $\alpha, \beta \in \overline{r+1, n}$ . (see [4], p. 58)

Extending the notion of distribution we obtain the definition of an *IDS* of a Lie algebroid. A characterization of the involutivity of an *IDS* in a result of Frobenius type is presented in *Theorem 4.7*.

This paper studies the intersection between the geometry of Lie algebroids and some aspects of *EDS*. In the classical sense, an *EDS* is a pair  $(M, \mathcal{I})$  consisting of a smooth manifold  $M$  and a homogeneous, differentially closed ideal  $\mathcal{I}$  in the algebra of smooth differential forms on  $M$ . (see [5, 6]) Using the notion of *EDS* of an arbitrary Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$  we obtained a new result of Cartan type in the *Theorem 5.1*.

In the particular case of standard Lie algebroid  $((TM, \tau_M, M), [\cdot]_{TM}, (Id_{TM}, Id_M))$  there are obtained similar results those for distributions.

We know that a submanifold  $S$  of  $N$  is said to be *integral manifold* for the distribution  $\mathcal{D}$  if for every point  $x \in N$ ,  $\mathcal{D}_x$  coincides with  $T_x S$ . The distribution  $\mathcal{D}$  is said to be *integrable* if for each point  $x \in N$  there exists an integral manifold of  $\mathcal{D}$  containing  $x$ .

As a distribution  $\mathcal{D}$  is involutive if and only if it is integrable, then the study of the integral manifolds of an *IDS* or *EDS* is a new direction by research.

## 2 Preliminaries

In general, if  $\mathcal{C}$  is a category, then we denote  $|\mathcal{C}|$  the class of objects and for any  $A, B \in |\mathcal{C}|$ , we denote  $\mathcal{C}(A, B)$  the set of morphisms of  $A$  source and  $B$  target. Let **Liealg**, **Mod**, and **B<sup>v</sup>** be the category of Lie algebras, modules and vector bundles respectively.

We know that if  $(E, \pi, M) \in |\mathbf{B}^v|$ ,  $\Gamma(E, \pi, M) = \{u \in \mathbf{Man}(M, E) : u \circ \pi = Id_M\}$  and  $\mathcal{F}(M) = \mathbf{Man}(M, \mathbb{R})$ , then  $(\Gamma(E, \pi, M), +, \cdot)$  is a  $\mathcal{F}(M)$ -module.

We know that a Lie algebroid is a vector bundle  $(F, \nu, N) \in |\mathbf{B}^v|$  so that there exists

$$(\rho, Id_N) \in \mathbf{B}^v((F, \nu, N), (TN, \tau_N, N))$$

and also an operation

$$\begin{array}{ccc} \Gamma(F, \nu, N) \times \Gamma(F, \nu, N) & \xrightarrow{[\cdot]_F} & \Gamma(F, \nu, N) \\ (u, v) & \longmapsto & [u, v]_F \end{array}$$

with the following properties:

$LA_1$ . the equality holds good

$$[u, f \cdot v]_F = f[u, v]_F + \Gamma(\rho, Id_N)(u) f \cdot v,$$

for all  $u, v \in \Gamma(F, \nu, N)$  and  $f \in \mathcal{F}(N)$ ,

$LA_2$ . the 4-tuple  $(\Gamma(F, \nu, N), +, \cdot, [, ]_F)$  is a Lie  $\mathcal{F}(N)$ -algebra,

$LA_3$ . the **Mod**-morphism  $\Gamma(\rho, Id_N)$  is a **LieAlg**-morphism of  $(\Gamma(F, \nu, N), +, \cdot, [, ]_F)$  source and  $(\Gamma(TN, \tau_N, N), +, \cdot, [, ]_{TN})$  target.

Let  $\left((F, \nu, N), [, ]_F, (\rho, Id_N)\right)$  be a Lie algebroid.

- Locally, for any  $\alpha, \beta \in \overline{1, p}$ , we set  $[t_\alpha, t_\beta]_F = L_{\alpha\beta}^\gamma t_\gamma$ . We easily obtain that  $L_{\alpha\beta}^\gamma = -L_{\beta\alpha}^\gamma$ , for any  $\alpha, \beta, \gamma \in \overline{1, p}$ .

The real local functions  $\left\{L_{\alpha\beta}^\gamma, \alpha, \beta, \gamma \in \overline{1, p}\right\}$  are called the *structure functions*.

- We assume that  $(F, \nu, N)$  is a vector bundle with type fibre the real vector space  $(\mathbb{R}^p, +, \cdot)$  and structure group a Lie subgroup of  $(\mathbf{GL}(p, \mathbb{R}), \cdot)$ . We denote  $(x^i, z^\alpha)$  the canonical local coordinates on  $(F, \nu, N)$ , where  $i \in \overline{1, n}$ ,  $\alpha \in \overline{1, p}$ .

Consider

$$(x^i, z^\alpha) \longrightarrow (x^{i'}, z^{\alpha'})$$

a change of coordinates on  $(F, \nu, N)$ . Then the coordinates  $z^\alpha$  change to  $z^{\alpha'}$  according to the rule:

$$(2.1) \quad z^{\alpha'} = \Lambda_\alpha^{\alpha'} z^\alpha.$$

- If  $z^\alpha t_\alpha \in \Gamma(F, \nu, N)$  is arbitrary, then

$$(2.2) \quad [\Gamma(\rho, Id_N)(z^\alpha t_\alpha) f](x) = \left(\rho_\alpha^i z^\alpha \frac{\partial f}{\partial x^i}\right)(x)$$

for any  $f \in \mathcal{F}(N)$  and  $x \in N$ .

The coefficients  $\rho_\alpha^i$  change to  $\rho_{\alpha'}^{i'}$  according to the rule:

$$(2.3) \quad \rho_{\alpha'}^{i'} = \Lambda_\alpha^{\alpha'} \rho_\alpha^i \frac{\partial x^{i'}}{\partial x^i},$$

where

$$\|\Lambda_\alpha^{\alpha'}\| = \left\|\Lambda_\alpha^{\alpha'}\right\|^{-1}.$$

The following equalities hold good:

$$(2.4) \quad \left(\rho_\alpha^i \frac{\partial}{\partial x^i}\right)(f) = \rho_\alpha^i \frac{\partial f}{\partial x^i}, \forall f \in \mathcal{F}(N).$$

and

$$(2.5) \quad L_{\alpha\beta}^\gamma \cdot \rho_\gamma^k = \rho_\alpha^i \cdot \frac{\partial \rho_\beta^k}{\partial x^i} - \rho_\beta^j \cdot \frac{\partial \rho_\alpha^k}{\partial x^j}.$$

### 3 Interior Differential Systems

Let  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$  be a Lie algebroid.

**Definition 3.1** Any vector subbundle  $(E, \pi, N)$  of the vector bundle  $(F, \nu, N)$  will be called *interior differential system (IDS) of the Lie algebroid*

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N)).$$

*Remark 3.1* If  $(E, \pi, M)$  is an IDS of the Lie algebroid

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N)),$$

then we obtain a vector subbundle  $(E^0, \pi^0, N)$  of the dual vector bundle  $\left(\overset{*}{F}, \overset{*}{\nu}, N\right)$  so that

$$\Gamma(E^0, \pi^0, N) \stackrel{put}{=} \left\{ \Omega \in \Gamma\left(\overset{*}{F}, \overset{*}{\nu}, N\right) : \Omega(S) = 0, \forall S \in \Gamma(E, \pi, N) \right\}.$$

The vector subbundle  $(E^0, \pi^0, N)$  will be called *the annihilator vector subbundle of the IDS  $(E, \pi, N)$* .

**Proposition 3.1** If  $(E, \pi, N)$  is an IDS of the Lie algebroid

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$$

so that  $\Gamma(E, \pi, N) = \langle S_1, \dots, S_r \rangle$ , then it exists  $\Theta^{r+1}, \dots, \Theta^p \in \Gamma\left(\overset{*}{F}, \overset{*}{\nu}, N\right)$  linearly independent so that

$$\Gamma(E^0, \pi^0, N) = \langle \Theta^{r+1}, \dots, \Theta^p \rangle.$$

**Definition 3.2** The IDS  $(E, \pi, N)$  of the Lie algebroid  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$  will be called *involutive* if  $[S, T]_F \in \Gamma(E, \pi, N)$ , for any  $S, T \in \Gamma(E, \pi, N)$ .

**Proposition 3.2** If  $(E, \pi, N)$  is an IDS of the Lie algebroid  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, \eta))$  and  $\{S_1, \dots, S_r\}$  is a base of the  $\mathcal{F}(M)$ -submodule  $(\Gamma(E, \pi, N), +, \cdot)$  then  $(E, \pi, N)$  is involutive if and only if  $[S_a, S_b]_F \in \Gamma(E, \pi, N)$ , for any  $a, b \in \overline{1, r}$ .

### 4 Exterior differential calculus

Let  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$  be a Lie algebroid.

We denote  $\Lambda^q(F, \nu, N)$  the set of *differential forms of degree  $q$* . If

$$\Lambda(F, \nu, N) = \bigoplus_{q \geq 0} \Lambda^q(F, \nu, N),$$

then we obtain *the exterior differential algebra*  $(\Lambda(F, \nu, N), +, \cdot, \wedge)$ .

**Definition 4.1** For any  $z \in \Gamma(F, \nu, N)$ , the application

$$\Lambda(F, \nu, N) \xrightarrow{L_z} \Lambda(F, \nu, N),$$

defined by

$$L_z(f) = [\Gamma(\rho, Id_N)z](f),$$

for any  $f \in \mathcal{F}(N)$  and

$$\begin{aligned} L_z \omega(z_1, \dots, z_q) &= [\Gamma(\rho, Id_N) z](\omega((z_1, \dots, z_q))) \\ &\quad - \sum_{i=1}^q \omega((z_1, \dots, [z, z_i]_F, \dots, z_q)), \end{aligned}$$

for any  $\omega \in \Lambda^q(F, \nu, N)$  and  $z_1, \dots, z_q \in \Gamma(F, \nu, N)$ , is called *the covariant Lie derivative with respect to the section  $z$* .

**Theorem 4.1** *If  $z \in \Gamma(F, \nu, N)$ ,  $\omega \in \Lambda^q(F, \nu, N)$  and  $\theta \in \Lambda^r(F, \nu, N)$ , then*

$$(4.1) \quad L_z(\omega \wedge \theta) = L_z \omega \wedge \theta + \omega \wedge L_z \theta.$$

**Definition 4.2** For any  $z \in \Gamma(F, \nu, N)$ , the application

$$\begin{aligned} \Lambda(F, \nu, N) &\xrightarrow{i_z} \Lambda(F, \nu, N) \\ \Lambda^q(F, \nu, N) \ni \omega &\longmapsto i_z \omega \in \Lambda^{q-1}(F, \nu, N), \end{aligned}$$

defined by  $i_z f = 0$ , for any  $f \in \mathcal{F}(N)$  and

$$i_z \omega(z_2, \dots, z_q) = \omega(z, z_2, \dots, z_q),$$

for any  $z_2, \dots, z_q \in \Gamma(F, \nu, N)$ , is called *the interior product associated to the section  $z$* .

**Theorem 4.2** *If  $z \in \Gamma(F, \nu, N)$ , then for any  $\omega \in \Lambda^q(F, \nu, N)$  and  $\theta \in \Lambda^r(F, \nu, N)$  we obtain*

$$(4.2) \quad i_z(\omega \wedge \theta) = i_z \omega \wedge \theta + (-1)^q \omega \wedge i_z \theta.$$

**Theorem 4.3** *For any  $z, v \in \Gamma(F, \nu, N)$  we obtain*

$$(4.3) \quad L_v \circ i_z - i_z \circ L_v = i_{[z, v]_F}.$$

**Theorem 4.4** *The application*

$$\begin{aligned} \Lambda^q(F, \nu, N) &\xrightarrow{d^F} \Lambda^{q+1}(F, \nu, N) \\ \omega &\longmapsto d\omega \end{aligned}$$

defined by

$$d^F f(z) = \Gamma(\rho, Id_N)(z) f,$$

for any  $z \in \Gamma(F, \nu, N)$ , and

$$\begin{aligned} d^F \omega(z_0, z_1, \dots, z_q) &= \sum_{i=0}^q (-1)^i \Gamma(\rho, Id_N) z_i(\omega((z_0, z_1, \dots, \hat{z}_i, \dots, z_q))) \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([z_i, z_j]_F, z_0, z_1, \dots, \hat{z}_i, \dots, \hat{z}_j, \dots, z_q), \end{aligned}$$

for any  $z_0, z_1, \dots, z_q \in \Gamma(F, \nu, N)$ , is unique having the following property:

$$(4.4) \quad L_z = d^F \circ i_z + i_z \circ d^F, \quad \forall z \in \Gamma(F, \nu, N).$$

This application is called *the exterior differentiation operator of the exterior differential algebra of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$* .



**Theorem 4.5** *The exterior differentiation operator  $d^F$  given by the previous theorem has the following properties:*

1. *For any  $\omega \in \Lambda^q(F, \nu, N)$  and  $\theta \in \Lambda^r(F, \nu, N)$  we obtain*

$$(4.5) \quad d^F(\omega \wedge \theta) = d^F\omega \wedge \theta + (-1)^q \omega \wedge d^F\theta.$$

2. *For any  $z \in \Gamma(F, \nu, N)$  we obtain*

$$(4.6) \quad L_z \circ d^F = d^F \circ L_z.$$

3.  $d^F \circ d^F = 0$ .

**Theorem 4.6** (of Maurer-Cartan type)

*If  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$  is a Lie algebroid and  $d^F$  is the exterior differentiation operator of the exterior differential  $\mathcal{F}(N)$ -algebra  $(\Lambda(F, \nu, N), +, \cdot, \wedge)$ , then we obtain the structure equations of Maurer-Cartan type*

$$(\mathcal{MC}_1) \quad d^F t^\alpha = -\frac{1}{2} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma, \quad \alpha \in \overline{1, p}$$

and

$$(\mathcal{MC}_2) \quad d^F x^i = \rho_\alpha^i t^\alpha, \quad i \in \overline{1, n},$$

where  $\{t^\alpha, \alpha \in \overline{1, p}\}$  is the coframe of the vector bundle  $(F, \nu, N)$ .

These equations will be called *the structure equations of Maurer-Cartan type associated to the Lie algebroid  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$ .*

*Proof.* Let  $\alpha \in \overline{1, p}$  be arbitrary. Since

$$d^F t^\alpha(t_\beta, t_\gamma) = -L_{\beta\gamma}^\alpha, \quad \forall \beta, \gamma \in \overline{1, p}$$

it results that

$$(1) \quad d^F t^\alpha = -\sum_{\beta < \gamma} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma.$$

Since  $L_{\beta\gamma}^\alpha = -L_{\gamma\beta}^\alpha$  and  $t^\beta \wedge t^\gamma = -t^\gamma \wedge t^\beta$ , for any  $\beta, \gamma \in \overline{1, p}$ , it results that

$$(2) \quad \sum_{\beta < \gamma} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma = \frac{1}{2} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma$$

Using the equalities (1) and (2) it results the structure equation  $(\mathcal{MC}_1)$ .

Let  $i \in \overline{1, n}$  be arbitrary. Since

$$d^F x^i(t_\alpha) = \rho_\alpha^i, \quad \forall \alpha \in \overline{1, p}$$

it results the structure equation  $(\mathcal{MC}_2)$ .

*q.e.d.*

**Remark 4.1** In the particular case of the standard Lie algebroid

$$((TN, \tau_N, N), [\cdot, \cdot]_{TN}, (Id_{TN}, Id_N))$$

we obtain

$$(\mathcal{MC}_2)' \quad d^{TN} x^i = dx^i, \quad i \in \overline{1, n},$$

where  $\{dx^i, i \in \overline{1, n}\}$  is the coframe of the vector bundle  $(TN, \tau_N, N)$ .

As  $d^{TN} \circ d^{TN} = 0$  and  $L_{jk}^i = 0$ , for all  $i, j, k \in \overline{1, n}$  we obtain

$$(\mathcal{MC}_1)' \quad d^{TN} (dx^i) = 0 = -\frac{1}{2} L_{jk}^i dx^j \wedge dx^k, \quad i \in \overline{1, n}$$

This equations are the structure equations of Maurer-Cartan type associated to the standard Lie algebroid  $((TN, \tau_N, N), [\cdot, \cdot]_{TN}, (Id_{TN}, Id_N))$ .

**Theorem 4.7** (of Frobenius type) *Let  $(E, \pi, N)$  be an IDS of the Lie algebroid*

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N)).$$

*If  $\{\Theta^{r+1}, \dots, \Theta^p\}$  is a base of the  $\mathcal{F}(N)$ -submodule  $(\Gamma(E^0, \pi^0, N), +, \cdot)$ , then the IDS  $(E, \pi, N)$  is involutive if and only if it exists*

$$\Omega_\beta^\alpha \in \Lambda^1(F, \nu, N), \quad \alpha, \beta \in \overline{r+1, p}$$

so that

$$d^F \Theta^\alpha = \sum_{\beta \in \overline{r+1, p}} \Omega_\beta^\alpha \wedge \Theta^\beta, \quad \alpha \in \overline{r+1, p}.$$

*Proof.* Let  $\{S_1, \dots, S_r\}$  be a base of the  $\mathcal{F}(N)$ -submodule  $(\Gamma(E, \pi, N), +, \cdot)$

Let  $\{S_{r+1}, \dots, S_p\} \in \Gamma(F, \nu, N)$  so that  $\{S_1, \dots, S_r, S_{r+1}, \dots, S_p\}$  is a base of the  $\mathcal{F}(N)$ -module

$$(\Gamma(F, \nu, N), +, \cdot).$$

Let  $\Theta^1, \dots, \Theta^r \in \Gamma\left(\overset{*}{F}, \overset{*}{\nu}, N\right)$  so that  $\{\Theta^1, \dots, \Theta^r, \Theta^{r+1}, \dots, \Theta^p\}$  is a base of the  $\mathcal{F}(N)$ -module

$$\left(\Gamma\left(\overset{*}{F}, \overset{*}{\nu}, N\right), +, \cdot\right).$$

For any  $a, b \in \overline{1, r}$  and  $\alpha, \beta \in \overline{r+1, p}$ , we have the equalities:

$$\begin{aligned} \Theta^a(S_b) &= \delta_b^a \\ \Theta^a(S_\beta) &= 0 \\ \Theta^\alpha(S_b) &= 0 \\ \Theta^\alpha(S_\beta) &= \delta_\beta^\alpha \end{aligned}$$

We remark that the set of the 2-forms

$$\left\{ \Theta^a \wedge \Theta^b, \Theta^a \wedge \Theta^\beta, \Theta^\alpha \wedge \Theta^\beta, \quad a, b \in \overline{1, r} \wedge \alpha, \beta \in \overline{r+1, p} \right\}$$

is a base of the  $\mathcal{F}(M)$ -module  $(\Lambda^2(F, \nu, N), +, \cdot)$ .

Therefore, we have

$$(1) \quad d^F \Theta^\alpha = \sum_{b < c} A_{bc}^\alpha \Theta^b \wedge \Theta^c + \sum_{b, \gamma} B_{b\gamma}^\alpha \Theta^b \wedge \Theta^\gamma + \sum_{\beta < \gamma} C_{\beta\gamma}^\alpha \Theta^\beta \wedge \Theta^\gamma,$$

where,  $A_{bc}^\alpha, B_{b\gamma}^\alpha$  and  $C_{\beta\gamma}^\alpha$ ,  $a, b, c \in \overline{1, r}$ ,  $\alpha, \beta, \gamma \in \overline{r+1, p}$  are real local functions so that  $A_{bc}^\alpha = -A_{cb}^\alpha$  and  $C_{\beta\gamma}^\alpha = -C_{\gamma\beta}^\alpha$ .

Using the formula

$$(2) \quad \begin{aligned} d^F \Theta^\alpha (S_b, S_c) &= \Gamma(\rho, Id_N) S_b (\Theta^\alpha (S_c)) - \Gamma(\rho, Id_N) S_c (\Theta^\alpha (S_b)) \\ &\quad - \Theta^\alpha ([S_b, S_c]_F), \end{aligned}$$

we obtain that

$$(3) \quad A_{bc}^\alpha = -\Theta^\alpha ([S_b, S_c]_F),$$

for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ .

We admit that  $(E, \pi, N)$  is an involutive *IDS* of the Lie algebroid  $((F, \nu, N), [, ]_F, (\rho, Id_N))$ .

As  $[S_b, S_c]_F \in \Gamma(E, \pi, N)$ , for any  $b, c \in \overline{1, r}$ , it results that  $\Theta^\alpha ([S_b, S_c]_F) = 0$ , for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ . Therefore, for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ , we obtain  $A_{bc}^\alpha = 0$  and

$$\begin{aligned} d^F \Theta^\alpha &= \Sigma_{b, \gamma} B_{b\gamma}^\alpha \Theta^b \wedge \Theta^\gamma + \frac{1}{2} C_{\beta\gamma}^\alpha \Theta^\beta \wedge \Theta^\gamma \\ &= \left( B_{b\gamma}^\alpha \Theta^b + \frac{1}{2} C_{\beta\gamma}^\alpha \Theta^\beta \right) \wedge \Theta^\gamma. \end{aligned}$$

As

$$\Omega_\gamma^\alpha \stackrel{put}{=} B_{b\gamma}^\alpha \Theta^b + \frac{1}{2} C_{\beta\gamma}^\alpha \Theta^\beta \in \Lambda^1(F, \nu, N),$$

for any  $\alpha, \beta \in \overline{r+1, p}$ , it results the first implication.

Conversely, we admit that it exists

$$\Omega_\beta^\alpha \in \Lambda^1(F, \nu, N), \quad \alpha, \beta \in \overline{r+1, p}$$

so that

$$(4) \quad d^F \Theta^\alpha = \Sigma_{\beta \in \overline{r+1, p}} \Omega_\beta^\alpha \wedge \Theta^\beta,$$

for any  $\alpha \in \overline{r+1, p}$ .

Using the affirmations (1), (2) and (4) we obtain that  $A_{bc}^\alpha = 0$ , for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ .

Using the affirmation (3), we obtain  $\Theta^\alpha ([S_b, S_c]_F) = 0$ , for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ .

Therefore, we have  $[S_b, S_c]_F \in \Gamma(E, \pi, N)$ , for any  $b, c \in \overline{1, r}$ . Using the *Proposition 3.2*, we obtain the second implication. *q.e.d.*

## 5 Exterior Differential Systems

Let  $((F, \nu, N), [, ]_F, (\rho, Id_N))$  be a Lie algebroid.

**Definition 5.1** Any ideal  $(\mathcal{I}, +, \cdot)$  of the exterior differential algebra of the Lie algebroid  $((F, \nu, N), [, ]_F, (\rho, Id_M))$  closed under differentiation operator  $d^F$ , namely  $d^F \mathcal{I} \subseteq \mathcal{I}$ , is called *differential ideal of the Lie algebroid*  $((F, \nu, N), [, ]_F, (\rho, Id_M))$ .

**Definition 5.2** Let  $(\mathcal{I}, +, \cdot)$  be a differential ideal of the Lie algebroid

$$((F, \nu, N), [, ]_F, (\rho, Id_M)).$$

If it exists an *IDS*  $(E, \pi, N)$  so that for all  $k \in \mathbb{N}^*$  and  $\omega \in \mathcal{I} \cap \Lambda^k(F, \nu, N)$  we have  $\omega(u_1, \dots, u_k) = 0$ , for any  $u_1, \dots, u_k \in \Gamma(E, \pi, N)$ , then we will say that  $(\mathcal{I}, +, \cdot)$  is an *exterior differential system (EDS) of the Lie algebroid*  $((F, \nu, N), [, ]_F, (\rho, Id_N))$ .

**Theorem 5.1** (of Cartan type) *The IDS  $(E, \pi, N)$  of the Lie algebroid*

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$$

*is involutive, if and only if the ideal generated by the  $\mathcal{F}(N)$ -submodule  $(\Gamma(E^0, \pi^0, N), +, \cdot)$  is an EDS of the Lie algebroid  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$ .*

*Proof.* Let  $(E, \pi, N)$  be an involutive IDS of the Lie algebroid

$$((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N)).$$

Let  $\{\Theta^{r+1}, \dots, \Theta^p\}$  be a base of the  $\mathcal{F}(N)$ -submodule  $(\Gamma(E^0, \pi^0, N), +, \cdot)$ .

We know that

$$\mathcal{I}(\Gamma(E^0, \pi^0, N)) = \cup_{q \in \mathbb{N}} \{\Omega_\alpha \wedge \Theta^\alpha, \{\Omega_{r+1}, \dots, \Omega_p\} \subset \Lambda^q(F, \nu, N)\}.$$

Let  $q \in \mathbb{N}$  and  $\{\Omega_{r+1}, \dots, \Omega_p\} \subset \Lambda^q(F, \nu, N)$  be arbitrary.

Using the *Theorems 4.5 and 4.7* we obtain

$$\begin{aligned} d^F(\Omega_\alpha \wedge \Theta^\alpha) &= d^F \Omega_\alpha \wedge \Theta^\alpha + (-1)^{q+1} \Omega_\beta \wedge d^F \Theta^\beta \\ &= (d^F \Omega_\alpha + (-1)^{q+1} \Omega_\beta \wedge \Omega_\alpha^\beta) \wedge \Theta^\alpha. \end{aligned}$$

As

$$d^F \Omega_\alpha + (-1)^{q+1} \Omega_\beta \wedge \Omega_\alpha^\beta \in \Lambda^{q+2}(F, \nu, N)$$

it results that

$$d^F(\Omega_\beta \wedge \Theta^\beta) \in \mathcal{I}(\Gamma(E^0, \pi^0, N))$$

Therefore,

$$d^F \mathcal{I}(\Gamma(E^0, \pi^0, N)) \subseteq \mathcal{I}(\Gamma(E^0, \pi^0, N)).$$

Conversely, let  $(E, \pi, N)$  be an IDS of the Lie algebroid  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$  so that the  $\mathcal{F}(N)$ -submodule  $(\mathcal{I}(\Gamma(E^0, \pi^0, N)), +, \cdot)$  is an EDS of the Lie algebroid  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$ .

Let  $\{\Theta^{r+1}, \dots, \Theta^p\}$  be a base of the  $\mathcal{F}(N)$ -submodule  $(\Gamma(E^0, \pi^0, N), +, \cdot)$ . As

$$d^F \mathcal{I}(\Gamma(E^0, \pi^0, N)) \subseteq \mathcal{I}(\Gamma(E^0, \pi^0, N))$$

it results that it exists

$$\Omega_\beta^\alpha \in \Lambda^1(F, \nu, N), \quad \alpha, \beta \in \overline{r+1, p}$$

so that

$$d^F \Theta^\alpha = \sum_{\beta \in \overline{r+1, p}} \Omega_\beta^\alpha \wedge \Theta^\beta \in \mathcal{I}(\Gamma(E^0, \pi^0, N)).$$

Using the *Theorem 4.7* there results that  $(E, \pi, N)$  is an involutive IDS. *q.e.d.*

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